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The role of negative energy waves in some instabilities of parallel flows

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Parallel flows with step function velocity and density profiles can support waves which have negative energy, in the sense that exciting them lowers the total energy of the system. A number of instabilities can occur because of the coexistence of positive and negative energy waves, or because of the damping of negative energy waves; some particular examples are discussed to show how appreciation of this role of negative energy waves allows one to predict the existence of instability before doing any detailed analysis, and to gain insight into the instability mechanism.

1. Introduction

The idea of negative energy waves is familiar in plasma physics and is of considerable value in the discussion of both linear and nonlinear instabilities of uniform media (Bekefi 1966; Briggs 1964; Coppi, Rosenbluth & Sudan 1969; Davidson 1972). A wave has negative energy if its establishment in a previously unperturbed system requires that energy be extracted from rather than fed into the system, or, in other words, exciting the waves lowers the total energy of the system. One of the features of a uniform plasma, which makes this concept very useful, is that the dispersion relation for plane electrostatic waves is

$$\epsilon(\omega, k) = 0,$$

where ϵ is the linear dielectric constant of the plasma, while the wave energy in the above sense is

$$\omega \frac{\partial \epsilon}{\partial \omega} \times \text{the electric field energy}$$

(Stix 1962). Thus, the wave energy can be found immediately from a linear analysis and used to predict the stability properties of the system.

Negative energy waves have been discussed occasionally in work on fluid mechanics (Landahl 1962; Benjamin 1963; Acheson 1976), but their role in predicting and elucidating the mechanism of certain types of instability has not been fully exploited. In this paper we shall consider inviscid parallel flows with step function velocity and density profiles, and show that methods similar to those familiar in the literature of plasma physics may be used to analyse and predict their behaviour.

In §2 we derive an expression for the energy of a wave on the system described above, the method being to consider the work done during an idealized process in which the wave is driven up by an external force applied on a surface in the fluid.

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This is essentially the procedure used in plasma physics and also by Landahl (1962). It is shown that a well-defined dispersion function $D(\omega, k)$ may be obtained, with the properties that the dispersion relation for waves is

$$D(\omega, k) = 0,$$

and the wave energy is

$$\tfrac{1}{4}(\omega\, \partial D/\partial\omega) \times (\text{wave amplitude})^2,$$

so that D plays a role analogous to that of the dielectric constant in the theory of plasma waves. The function D, and hence the energy of a given mode, depends on the frame of reference from which the system is observed, and the energy can always be made negative by transforming to a suitable moving frame of reference. However, physically significant effects occur if both positive and negative energy modes exist in a given frame. Many linear instabilities occur through the coalescence of a positive and a negative energy mode to form a single unstable mode, such instabilities being known in plasma physics as reactive instabilities, or, in the classification of Benjamin (1963), as class C instabilities. In § 3 we discuss the Kelvin–Helmholtz instability and confirm that it is of this type, while in § 5 we consider a system with three fluid layers and show that such instabilities can occur when a positive energy wave propagates on one interface and a negative energy wave on the other. This example illustrates one of the useful features of this type of analysis, whereby the properties of a multi-layer system can be predicted by looking at each interface separately. If the interfaces are sufficiently far apart, waves on them are essentially independent, except when two of them happen to have the same frequency and wavenumber. Instability then occurs if the wave energies have opposite sign. In the three-layer system it will be shown that a density discontinuity at the lower interface introduces a wave mode which can interact with a negative energy wave on the upper interface and give rise to instability for flow velocities of the upper fluid below the critical velocity for Kelvin–Helmholtz instability. Similar instabilities were found by Taylor (1931). The method of this section can be extended to a system with an arbitrary number of interfaces, the technique being to plot the dispersion curve for waves on each interface separately, then to look for frequencies and wavenumbers through which two such curves pass. Instability occurs if the waves involved have energies of opposite sign. In this way the complex wave propagation and instability properties of such a system can be simplified and elucidated.

Another type of instability, class A in Benjamin's classification, occurs when in some frame of reference a wave has negative energy and there exists a dissipative process which extracts energy from the wave. If the dissipation is weak enough to leave the wave properties essentially unchanged, the extraction of energy from a negative energy wave leads to its growth and hence to instability. A simple example is discussed in § 4, where we consider an inviscid fluid flowing over a heavier, slightly viscous fluid at rest. The effect of the viscosity is clearly to extract energy from this system and, as would be expected, it produces instability if the flow velocity of the upper fluid is such as to give rise to a negative energy mode on the interface. As has already been pointed out by Weissman (1970) this produces an instability for flow velocities below the critical velocity for Kelvin–Helmholtz instability so that viscosity has a destabilizing effect. Essentially the same effect has been discussed by Landahl

(1962), the dissipation in this case being provided by damping in a flexible wall. Both Landahl and Weissman recognized that the physical mechanism for instability lies in the fact that the wave on the interface has negative energy. The present work yields a simple method of identifying such negative energy modes which, coupled with an awareness of their role, allows the prediction of such instabilities in advance of any detailed analysis of the dissipative system.

Finally, in § 6, we consider the weakly nonlinear theory of resonant wave interactions and discuss the conditions under which a nonlinear instability, of the type known in plasma physics as an explosive instability, can occur. This instability involves a resonant interaction amongst three waves, one with different sign of energy from the others, and leads to the simultaneous growth of all three. The necessary conditions are shown to be satisfied by some three-layer systems.

Some of the properties of negative energy waves which we discuss are implicit in the work of Landahl (1962), Benjamin (1963) and others. The points which we wish to emphasize here are the ease with which such waves can be identified in the class of flows considered, and the way in which such identification reveals sources of instability. The specific systems analysed are chosen to illustrate these ideas, rather than for their intrinsic importance.

2. Energy of waves on parallel flows

Consider an incompressible inviscid fluid whose velocity is in the x direction and may, along with the density, be a step-function of z. The fluid may be bounded, semi-infinite or infinite in the z direction. We shall look at waves propagating in the x direction though a y component of wavenumber could be introduced without adding any essential difficulty. Consider now a surface $z = z_0$ in the unperturbed fluid and suppose that when the wave is set up it is displaced to

$$z = z_0 + \eta(x,t),$$

where

$$\eta(x,t) = A \exp(ikx - i\omega t). \quad (1)$$

Linearization of the equations describing the fluid and substitution of perturbations going as $\exp(ikx - i\omega t)$ will give the dispersion relation for the fluid. The value of A in (1) then fixes the amplitude of the waves.

Suppose that in the linearized theory the equations are solved for $z > z_0 + \eta$ and that when the appropriate boundary conditions at the upper limit of z are imposed the pressure at $z = z_0 + \eta$ is found to first order in the amplitude. This will be proportional to A and of the form

$$p_1(x,t) = D_1(\omega,k)\,A \exp(ikx - i\omega t).$$

A similar treatment of the region $z < z_0 + \eta$ will give the pressure at $z_0 + \eta$ in the form

$$p_2(x,t) = D_2(\omega,k)\,A \exp(ikx - i\omega t).$$

The perturbation has been assumed to be the same in both regions, so that the continuity condition across the surface is satisfied automatically. The other condition to be satisfied is that pressure is continuous, which gives

$$D_1(\omega,k) = D_2(\omega,k)$$

or

$$D(\omega,k) = D_1(\omega,k) - D_2(\omega,k) = 0. \quad (2)$$

Equation (2) is the dispersion relation describing the properties of waves in the linear approximation. In view of the fact that we shall later take the surface $z = z_0$ to correspond to the boundary between two fluids, we shall require that the surface tension force is included in either D_1 or D_2, that is we equate pressures either just below or just above the interface between the fluids. This will avoid the introduction of a separate surface tension term in the equations.

Now suppose that for some wavenumber k_0 there exists a solution ω_0 of (2), where ω_0 is taken to be real. (The definition of wave energy for waves which are not marginally stable or close to marginal stability presents problems.) To calculate the energy of this wave we suppose that it is driven up by imposing a suitable external driving force on the surface $z = z_0 + \eta$ and calculate the work done on the fluid. The surface perturbation will be taken to be

$$\eta(x,t) = A(t)\exp(ik_0x - i\omega_0 t),$$

where $A \to 0$ as $t \to -\infty$ and $A \to A_0$ as $t \to \infty$. Assuming the time variation of A to be slow compared to that of the exponential the pressures at the surface will be

$$p_1(x,t) \approx D_1(\omega_0 + i\,\partial/\partial t, k_0)\,A(t)\exp(ik_0x - i\omega_0 t)$$

and

$$p_2(x,t) \approx D_2(\omega_0 + i\,\partial/\partial t, k_0)\,A(t)\exp(ik_0x - i\omega_0 t),$$

where the partial derivative $\partial/\partial t$ only acts on the slowly varying amplitude. The rate at which work is done per unit area in the x, y plane by the external driving force is

$$-\dot{\eta}(p_1 - p_2) \approx i\omega_0\eta(p_1 - p_2). \tag{3}$$

In (3) we neglect the contribution to $\dot{\eta}$ due to the rate of change of amplitude, since it is small compared to $-i\omega_0\eta$. Also we note that $p_1 - p_2$ is not zero, since the oscillation with time dependent amplitude is not a normal mode of the system. The pressure difference across the surface when the oscillation is being driven with increasing amplitude is precisely the reason why work must be done on the system to establish the wave.

If we use the approximation

$$D_{1,2}(\omega_0 + i\,\partial/\partial t, k_0) \approx i[\partial D_{1,2}(\omega_0, k_0)/\partial\omega_0]\,\partial/\partial t,$$

then we have

$$p_1 - p_2 \approx i\frac{\partial D}{\partial\omega_0}\frac{dA}{dt}\exp(ik_0x - i\omega_0 t).$$

Substituting this in (3) and averaging over the fast-time scale of the oscillations we obtain the rate at which work is done as

$$\frac{dW}{dt} = \tfrac{1}{4}\omega_0\frac{\partial D}{\partial\omega_0}\frac{d}{dt}|A|^2,$$

and so that total work done in setting up the wave is

$$W = \tfrac{1}{4}\omega_0\frac{\partial D}{\partial\omega_0}|A_0|^2. \tag{4}$$

Thus, we have a simple expression for the wave energy, in terms of a well-defined form of the linear dispersion relation, which is analogous to the well-known expression for the energy of plasma waves discussed in § 1.

Our derivation of wave energy in this section is an adaptation of the method of Coppi *et al.* (1969) for plasma waves. A similar expression was obtained by Landahl (1962) in a discussion of flow over a flexible surface. In the following sections we shall see that negative energy waves play a role in linear and nonlinear instabilities other than those associated with a flexible wall.

3. Kelvin–Helmholtz instability

Consider a system whose unperturbed state consists of fluid (incompressible and inviscid) of density ρ_1 in the region $z > 0$, moving with uniform velocity U in the x direction. In the region $z < 0$ there is fluid of density $\rho_2 > \rho_1$ at rest, while gravity g acts in the negative z direction. If the displacement of the surface is

$$\eta(x,t) = A\cos(kx-\omega t),$$

then the velocity potentials are, in the upper region,

$$\phi_1 = ((\omega - kU)/k)\,A\,e^{-kz}\sin(kx-\omega t)$$

and, in the lower region,

$$\phi_2 = (\omega/k)\,A\,e^{kz}\sin(kx-\omega t).$$

The use of real quantities instead of complex exponentials perhaps makes the averaging processes slightly clearer, but makes no essential difference to the theory.

If the surface tension is γ, then the pressure difference across the surface $z = \eta$ (including the effect of surface tension discussed above) is

$$\left\{(\rho_1-\rho_2)\,g - k^2\gamma + \frac{1}{k}(\omega-kU)^2\rho_1 + \frac{\omega^2}{k}\rho_2\right\}A\cos\omega t.$$

Thus the function $D(\omega,k)$ of § 2 is given by

$$D(\omega,k) = (\rho_1-\rho_2)\,g - k^2\gamma + \frac{1}{k}(\omega-kU)^2\rho_1 + \frac{\omega^2}{k}\rho_2, \tag{5}$$

and the energy per unit area in a stable wave is

$$\tfrac{1}{4}\omega\frac{\partial D}{\partial\omega}|A|^2.$$

We can verify this fact by making a direct calculation of the difference between the energy of the system when the wave is present and that of the unperturbed system. The contribution due to kinetic energy is (per unit area in the x, y plane)

$$\tfrac{1}{2}\rho_2\int_{-\infty}^{A\cos(kx-\omega t)} 2\omega^2A^2\cos^2(kx-\omega t)\,e^{2kz}dz + \tfrac{1}{2}\rho_1\int_{A\cos kx-\omega t}^{\infty}\{[U+(\omega-ku)\,Ae^{-kz}$$
$$\times\cos(kx-\omega t)]^2 + (\omega-ku)^2A^2e^{-2kz}\cos^2(kx-\omega t) - U^2\}\,dz,$$

which is, to second order in A and averaged over the oscillations,

$$\tfrac{1}{4}\rho_2\frac{\omega^2}{k}A^2 + \frac{1}{4}\frac{\rho_1}{k}[(\omega-kU)^2 + 2kU(\omega-kU)]\,A^2. \tag{6}$$

Consider now the gravitational potential energy. In the upper region the displacement of a fluid element in the z direction is

$$A \cos(kx - \omega t)\, e^{-kz}, \tag{7}$$

while its position along the x axis is

$$x = x_0 + Ut + A\, e^{-kz} \sin(kx - \omega t), \tag{8}$$

where x_0 is a constant. Substituting (8) in (7), expanding to second order in A and averaging gives the average displacement as $-\frac{1}{2}kA^2 e^{-2kz}$. Multiplying by $\rho_1 g$ and integrating over z gives the change in potential energy of the upper fluid due to the wave as $-\frac{1}{4}\rho_1 g A^2$. Taking the lower fluid into account in the same way we obtain for the gravitational potential energy per unit area

$$\tfrac{1}{4}g(\rho_2 - \rho_1)\, A^2. \tag{9}$$

Finally the contribution due to surface tension is

$$\tfrac{1}{4}\gamma k^2 A^2 \tag{10}$$

and the total wave energy per unit area is the sum of (6), (9) and (10).

Using the fact that waves satisfy the dispersion relation $D(\omega, k) = 0$, with D given by (5), we obtain the energy in the form

$$\tfrac{1}{2}\rho_2 \frac{\omega^2}{k} A^2 + \frac{1}{2}\frac{\rho_1}{k}[(\omega - kU)^2 + kU(\omega - kU)]\, A^2 = \omega \left[\frac{1}{2}\frac{\rho_2 \omega}{k} + \frac{1}{2}\frac{\rho_1}{k}(\omega - kU)\right] A^2. \tag{11}$$

This is easily verified to be equal to $\frac{1}{4}\omega A^2\, \partial D/\partial\omega$ as predicted by the general theory of the last section. It can be seen that the only term which can give rise to negative energy is the second one within the square brackets in equation (6), which is only non-zero if U is non-zero. It arises because oscillations in flow velocity are in phase with oscillations in depth and will be negative if the depth of fluid is a maximum when the flow velocity is a minimum. Equipartition of kinetic and potential energy only occurs if $U = 0$.

Solving the dispersion relation gives

$$\omega = \frac{\rho_1}{\rho_1 + \rho_2} kU \pm \left\{\frac{C_0^2}{U^2} - \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)^2}\right\}^{\frac{1}{2}} kU, \tag{12}$$

where

$$C_0^2 = \left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)\frac{g}{k} + \frac{k\gamma}{\rho_2 + \rho_1}, \tag{13}$$

and if (12) is substituted into (11) we obtain, for the wave energy,

$$W = \pm\, \omega(\rho_1 + \rho_2)\, kU \left\{\frac{C_0^2}{U^2} - \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)^2}\right\}^{\frac{1}{2}}, \tag{14}$$

the sign corresponding to that taken in (12). From (14) we see that the energy is negative if the frequency of the mode corresponding to the negative sign in (12) becomes positive (or vice versa if $U < 0$). In figure 1 the dispersion diagram for this system is shown, for the case where U is large enough to excite the Kelvin–Helmholtz instability. The mode labelled 2 is of negative energy in the region where its frequency is positive, the reason being that the total kinetic energy of the system is less than that

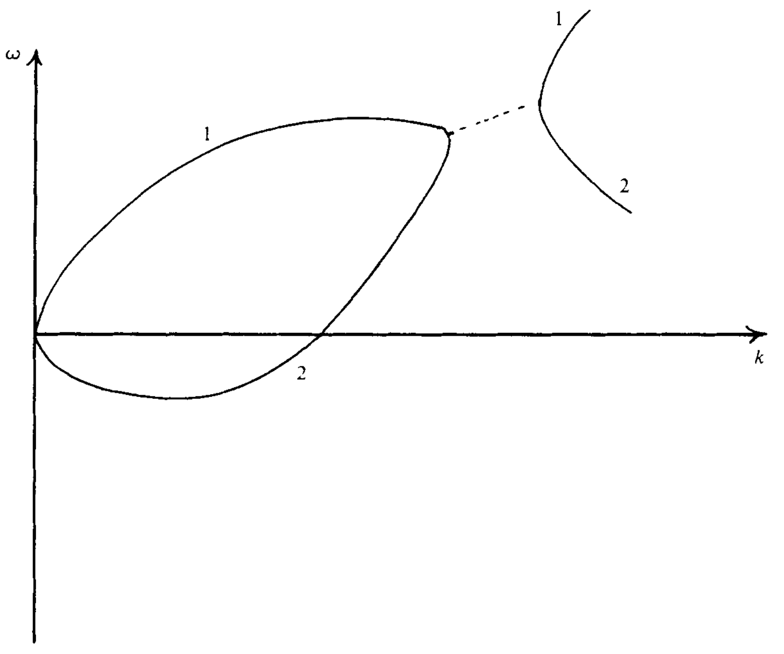

FIGURE 1. Dispersion curves for Kelvin–Helmholtz instability. – – –, unstable (ω complex).

of the unperturbed system. The unstable region is produced by a coalescence of positive and negative energy modes, a feature characteristic of a large class of instabilities [reactive instabilities in the terminology of plasma physics (Bekefi 1966)]. Note that there may be a negative energy wave even if U is less than the critical velocity for Kelvin–Helmholtz instability.

A change in the frame of reference may change the sign of the energy in a given mode, but the unstable branch always occurs as above, except in the special case in which the modes coalesce at frequency zero. In this case the energy of each mode tends to zero at the point of coalescence. In a later section we shall illustrate how instabilities of this kind can be predicted in more complex systems by looking for points in the dispersion diagram where positive and negative energy modes meet.

4. The effect of viscosity

Suppose that in some frame of reference a negative energy wave exists and that some dissipation is introduced, small enough not to effect any large change in the wave propagation properties. If energy is lost by the negative energy wave then it grows in amplitude and is unstable. We illustrate this effect by assuming that in the system of the previous section the lower fluid has kinematic viscosity ν, while the upper fluid remains inviscid. Since the lower fluid is at rest in the unperturbed state the effect of viscosity in it must be to extract energy from the system so that we might expect a negative energy mode as described above to become unstable. This may happen at flow velocities of the upper fluid below the critical velocity for Kelvin–Helmholtz instability.

The problem of surface waves on a viscous fluid has been considered by Lamb (1906) (see also Miles 1959). Using his result and taking into account the effect of the upper fluid we obtain the dispersion relation

$$-(\rho_2-\rho_1)g-k^2\gamma+k^{-1}(\omega-kU)^2\rho_1+\rho_2 k^{-1}(\omega^2+4i\nu\omega k^2) = 0$$

or

$$D(\omega,k) = -4i\rho_2\nu\omega k. \tag{15}$$

Since we are only interested in the case where the viscous correction to the dispersion relation is small we have neglected terms of higher order in ν which appear in Lamb's expression. It is easily verified that if these terms are of the same order as that which we include then the term on the right-hand side of (15) is of the same order as the inertial terms on the left-hand side.

If ω_0 is a root of

$$D(\omega,k) = 0,$$

then for small ν we may take

$$\omega = \omega_0+\delta\omega,$$

where $\delta\omega$ is a small correction. Then we have

$$\delta\omega\,\partial D/\partial\omega_0 \approx -4i\rho_2\nu\omega_0 k$$

or

$$\delta\omega \approx -\frac{4i\rho_2\nu\omega_0 k}{\partial D/\partial\omega_0}. \tag{16}$$

Instability will occur if the imaginary part of $\delta\omega$ is positive, which can be seen, from (16), to occur if $\omega_0\,\partial D/\partial\omega_0 < 0$, that is the wave in question has negative energy.

This substantiates our claim that a negative energy wave may be driven unstable by the effect of viscosity, an effect similar to the class A instability of Benjamin (1963). For the system considered here it is easy to see that the negative energy wave can occur and be unstable for flow velocities below the critical velocity for the Kelvin–Helmholtz instability. Miles (1959) did not find this because of his assumption that, for Kelvin–Helmholtz instability, the pressure of the upper fluid was always in phase with the displacement, a result which is only true, for the system we are considering, if ω is real. If both fluids were viscous more care would have to be exercised. Quite apart from the fact that the unperturbed flow would be different, a wave with negative energy in the rest frame of the lower fluid could have positive energy in the rest frame of the upper fluid and be damped by it. Even if our analysis were relevant to real fluids, it should be noted that the growth rates below the critical velocity for Kelvin–Helmholtz instability would be very small in systems where our analysis is valid, so that our results are not necessarily in conflict with the experimental results of Francis (1956). The main purpose of our example is to illustrate the fact that it is possible to predict the consequence of small dissipative effects for a wave if the sign of its energy is known.

The effect of viscosity in producing instability at lower flow velocities than are required for Kelvin–Helmholtz instabilities has been considered by Weissman (1970), and the similar problem of destabilization due to a flexible damped wall by Landahl (1962). The main point of our discussion is not the originality of the result, but the fact that it is an obvious consequence of the fact that one of the waves on the interface is of negative energy. The negative energy wave is just that one whose direction of propagation is reversed by the effect of the flow of the upper fluid.

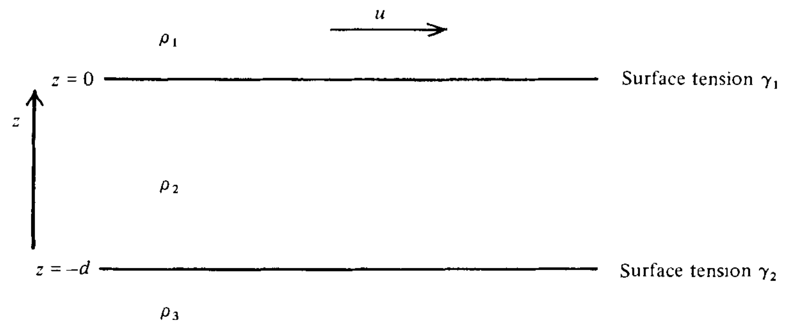

FIGURE 2. The unperturbed system with three fluid layers.

5. Stability of a three-layer system

We now consider the stability of a system as shown in figure 2. The lower two layers have been assumed to have no relative motion, though such motion could be introduced at the expense of a slight complication in the algebra. The velocity potentials in the three regions are of the form

$$\left.\begin{aligned}\phi_1 &= C_1 \exp(-kz)\exp(ikx-i\omega t),\\ \phi_2 &= [C_2\exp(-kz)+C_3\exp(kz)]\exp(ikx-i\omega t),\\ \phi_3 &= C_4\exp[k(z+d)]\exp(ikx-i\omega t).\end{aligned}\right\} \quad (17)$$

If the surface displacements at $z = 0$ and $z = -d$ are

$$\eta_1 = A_1 \exp(ikx-i\omega t) \quad \text{and} \quad \eta_2 = A_2\exp(ikx-i\omega t)$$

respectively, then the kinematic boundary conditions lead to the equations

$$\left.\begin{aligned}C_1 = i(\omega-kU)A_1/k,\quad C_2 = (-i\omega A_1+i\omega A_2 e^{kd})/k(e^{2kd}-1),\\ C_3 = (-i\omega A_1+i\omega A_2 e^{-kd})/k(1-e^{-2kd}) \quad\text{and}\quad C_4 = -i\omega A_2/k.\end{aligned}\right\} \quad (18)$$

The boundary conditions on the pressure then lead to the equations

$$\left.\begin{aligned}D_1(\omega,k)A_1+\rho_2(\omega^2/k)\operatorname{cosech}(kd)A_2 = 0\\ \text{and}\qquad D_2(\omega,k)A_2+\rho_2(\omega^2/k)\operatorname{cosech}(kd)A_1 = 0,\end{aligned}\right\} \quad (19)$$

where $$D_1(\omega,k) = (\rho_2-\rho_1)g+k^2\gamma_1-\frac{1}{k}[\rho_1(\omega-kU)^2+\rho_2\omega^2\coth(kd)]$$

and $D_1(\omega,k) = 0$ is the dispersion relation for waves on the upper interface if the lower one is replaced by a rigid boundary. Similarly $D_2(\omega,k) = 0$ is the dispersion relation for waves on the lower interface if the upper one is replaced by a rigid boundary, the form of D_2 being analogous to that of D_1. Note that the functions $D_{1,2}$ used here are not related to those of § 2.

Following our procedure of § 2 we can consider the pressure difference across the surface $z = 0$ when all other boundary conditions are satisfied. This is, using the second equation of (19) to eliminate A_2,

$$D(\omega,k)A_1 \equiv -\{D_1(\omega,k)-\rho_2^2\omega^4\operatorname{cosech}^2 kd/k^2D_2(\omega,k)\}A_1$$

and the wave energy is, as before,

$$\tfrac{1}{4}\omega\frac{\partial D}{\partial\omega}|A_1|^2.$$

If desired the wave energy could, of course, be expressed in terms of the amplitude A_2 or, indeed, of the amplitude at an arbitrary value of z.

The most interesting case for our present purposes occurs when $\operatorname{cosech}^2(kd) \ll 1$, in which case the waves on the two interfaces are only weakly coupled. Then, from the dispersion relation for waves on the complete system, i.e.

$$D(\omega, k) = 0$$

or

$$D_1(\omega, k)\, D_2(\omega, k) - (\rho_2^2\,\omega^4/k^2)\operatorname{cosech}^2(kd) = 0 \tag{20}$$

it is evident that, so long as the roots of $D_1 = 0$ and $D_2 = 0$ are well apart, the wave frequencies are close to those of waves on the interfaces considered separately. Also, it can be verified that the wave energies are of the same sign as those of the corresponding waves on a single interface.

Now let us consider what happens if, for some value of k, the roots of $D_1(\omega, k) = 0$ and of $D_2(\omega, k) = 0$ are close. In particular we suppose that

$$D_1(\omega_1, k) = 0 \quad \text{and} \quad D_2(\omega_2, k) = 0,$$

where $\omega_2 = \omega_1 + \delta$ and δ is small. Now, suppose that the solution of the complete dispersion relation (20) is

$$\omega = \omega_1 + \Delta,$$

where Δ is also small. Then we have

$$D_1(\omega_1 + \Delta, k)\, D_2(\omega_2 + \Delta - \delta, k) - (\rho_2^2\,\omega^4/k^2)\operatorname{cosech}^2(kd) = 0$$

or, approximately,

$$\Delta\frac{\partial D_1}{\partial\omega_1}(\Delta - \delta)\frac{\partial D_2}{\partial\omega_2} - \frac{\rho_2^2\,\omega_1^4}{k^2}\operatorname{cosech}^2(kd) = 0.$$

This is a quadratic equation for Δ of the form

$$\Delta^2 - \Delta\delta - K = 0,$$

where K is positive if the energy of both waves is of the same sign, and negative if one is of positive and the other of negative energy. From the solution of this it can be seen that in the latter case Δ has complex roots, leading to instability if

$$|\delta| < 2|K|.$$

This illustrates the fact that it is possible to predict instabilities of a system of this sort simply by looking at the dispersion properties of waves on each interface taken separately. If dispersion curves for the individual interfaces cross as shown in figure 3 (*a*), then the behaviour of the complete system is as in figure 3 (*b*) if the wave energies are both positive or both negative, or as in figure 3 (*c*) if one is positive and one negative.

This example illustrates a general principle which is of considerable value in the linear stability analysis of complex systems. If the system supports a number of different wave modes which only interact weakly with each other, then it may be

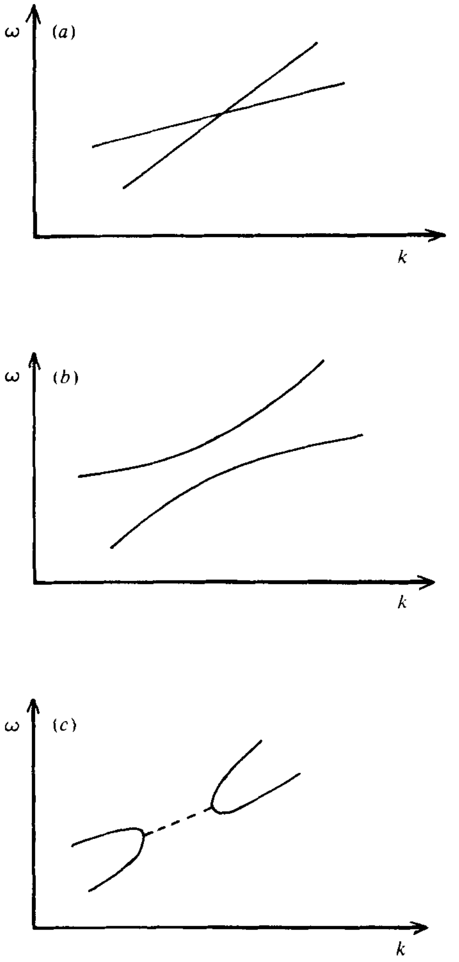

FIGURE 3. Form of the dispersion curves when modes on the upper and lower interfaces coincide: (*a*) for the interfaces treated separately; (*b*) for the complete system when the sign of the energy of the two modes is the same; (*c*) for the complete system when there is one positive and one negative energy mode. ---, unstable.

possible to investigate the stability by breaking the problem down into a number of simpler ones and looking for points in the dispersion diagram at which two modes intersect. For systems more complex than that considered here this idea may be very useful [for applications to plasma physics see Briggs (1964)]. To return to our particular example, it is of interest to note that if the density of the lowest fluid is close enough to, but not equal to, that of the middle fluid, then the system may be unstable for velocities of the upper fluid too low to excite the usual Kelvin–Helmholtz instability, though the growth rates will be small in systems for which our analysis is valid.

Some problems very similar to that considered here have been treated by Taylor (1931), the main difference being that he considered a linear, rather than a piecewise constant, velocity profile. Since the velocity potentials of perturbations of such a flow are still of the same form as (17), this makes very little difference to the analysis.

One conclusion of his work was that it was 'curious that the effect of a stratification in density gives rise to unstable waves for certain velocities when the same waves in a homogeneous fluid would be stable'. Like the instability discussed here, Taylor's instabilities occur when the backward-moving wave at the upper surface, which has negative energy, has the same frequency and wavenumber as a positive energy wave in the stratified fluid below this surface.

6. Nonlinear theory

If three wave modes exist in a system and form a resonant triad, that is their frequencies and wavenumbers satisfy the conditions

$$\omega_1 = \omega_2 + \omega_3, \quad \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3, \tag{21}$$

then, as is well known, their amplitudes obey a set of equations of the form

$$\frac{\partial A_1}{\partial t} = C_1 A_2 A_3, \quad \frac{\partial A_2}{\partial t} = C_2 A_1 A_3^* \quad \text{and} \quad \frac{\partial A_3}{\partial t} = C_3 A_1 A_2^*. \tag{22}$$

If the wave amplitudes are normalized so that the wave energy density is

$$\omega_i S_i |A_i^2|, \quad i = 1, 2, 3,$$

where S_i is ± 1 and is chosen so that the sign of the energy is correct, then the coefficients C_i obey, in a non-dissipative system, certain symmetry relations, from which follow various conservation relations, including energy conservation and the so-called Manley–Rowe relations

$$\frac{d}{dt}\{S_1|A_1|^2 + S_2|A_2^2|\} = \frac{d}{dt}\{S_1|A_1^2| + S_3|A_3^2|\}$$

$$= \frac{d}{dt}\{S_2|A_2^2| - S_3|A_3^2|\} = 0.$$

These relations and other consequences of the nonlinear coupling equations are discussed in detail for plasmas by Davidson (1972), and have been shown to hold for any Lagrangian system by Dougherty (1970).

A particularly interesting consequence (Coppi *et al.* 1969; Davidson 1972) is the possibility of solutions in which all the wave amplitudes grow simultaneously and tend to infinity at some finite time. For this to occur two conditions must be satisfied, namely that one of the waves has an energy which is different in sign from the other two and that this wave has the highest frequency in absolute value. Explosive instabilities of this type are well known in the theory of plasmas. In view of the fact that systems of the type we have discussed in previous sections can be described by a Lagrangian (Simmons 1969), it follows that we may expect explosive instability to occur in them if resonant triads satisfying the appropriate conditions can be found. In a system with a single interface, as discussed in § 3, stable positive and negative energy waves can occur for a range of velocities below the critical velocity for Kelvin–Helmholtz instability. However, it is easy to see from the dispersion relation that triads satisfying the conditions for explosive instability do not exist. If, however, we go to a system with three fluid layers, as discussed in § 5, then such a triad can exist.

To see this, suppose that the velocity U is large enough to produce a negative energy wave of frequency ω_1 and wavenumber k_1 in the upper interface, but is below the critical velocity for Kelvin–Helmholtz instability. On the lower interface all modes have positive energy. Neglecting surface tension the dispersion relation for these modes is

$$\omega = Ak^{\frac{1}{2}}, \tag{23}$$

where A depends on the ratio of the densities of the two lower fluids. Frequencies and wavenumbers satisfying (23) and

$$\omega_1 = \omega_2 + \omega_3, \quad k_1 = k_2 + k_3$$

can be found if

$$2A^2k_1 \geqslant \omega_1^2 > A^2k_1. \tag{24}$$

Thus, given the properties of the upper interface, explosively unstable resonant triads will exist if A is small enough, that is if the density jump across the lower interface is sufficiently small. This analysis assumes that the interfaces are well apart, so the nonlinear coupling would be expected to be rather weak. Detailed analysis of the full dispersion relation of § 5 would be required to see if the explosive instability persists when the middle fluid layer is thinner. Also, it may be noted that if (24) is satisfied there exists a weak linear instability of the type discussed in § 5, though, in general, at a different wavenumber from the members of the explosively unstable triad. It is perhaps worth pointing out here that, although the sign of the energy of a particular wave may change when viewed from a different frame of reference, the conditions for existence of an explosive instability are satisfied in all frames of reference if they are satisfied in one (Davidson 1972). The mode which is of different energy sign and highest frequency may, however, be different in different frames of reference.

The explosive instability discussed here is different in origin from the similar phenomenon discussed by Craik (1968). The system described in that work is dissipative and the coupling coefficients in the nonlinear equations do not, in consequence, obey the symmetry relations mentioned earlier. On a somewhat cautionary note it should be mentioned that, although explosive instabilities are well known in the theory of plasmas, only very recently has experimental evidence of their existence been obtained (Sugaya, Sugawa & Nomoto 1977). One possible explanation is that higher order effects may stabilize the system at a comparatively low wave amplitude level. It might, therefore, be of some interest to look for experimentally accessible fluid systems which should, in theory, exhibit explosive instability.

The nonlinear coupling coefficients appropriate to the three layer system which we have described above have, recently, been calculated by A. D. D. Craik & J. Adam (private communication). Their results have verified the predictions on the symmetry of the coupling coefficients and of the existence of the explosive instability as described above.

7. Conclusion

We have shown that the energy of waves on inviscid, incompressible parallel flows with step-function velocity and density profiles may be obtained easily from a standard linear analysis of the problem, the formula being analogous to that which is well known and has proved to be useful in plasma physics. A number of simple systems have been analysed in order to verify this formula and to show that identification of negative energy waves may be useful in investigating both the linear and nonlinear behaviour of the system.

In particular, three types of problem in which this identification is useful were illustrated. First the effect of a weak dissipative process on a wave which is neutrally stable according to inviscid theory was considered, and it was shown that this could be predicted, qualitatively, from a knowledge of the sign of the wave energy. Then it was demonstrated that instability of a complex system could, in some cases, be predicted by looking at the properties of waves on simpler component systems and seeking points on the dispersion diagram at which positive and negative energy modes intersect. Finally, the possibility of nonlinear instability was discussed, and it was pointed out that such instability can occur if a certain type of resonant triad, containing both positive and negative energy modes, exists.

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